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Darboux related quantum integrable systems on a constant curvature surface

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Abstract

We consider integrable deformations of the Laplace–Beltrami operator on a constant curvature surface, obtained through the action of first-order Darboux transformations. Darboux transformations are related to the symmetries of the underlying geometric space and lead to separable potentials which are related to the KdV equation. Eigenfunctions of the corresponding operators are related to highest weight representations of the symmetry algebra of the underlying space.

Keywords: Quantum integrability; Laplace–Beltrami; Darboux transformations

1. Introduction

In this paper we consider the Laplace–Beltrami operator of a space of constant curvature and build integrable potential functions through Darboux transformations related to the symmetries of the underlying geometric space. We start by recalling some basic geometric facts.

For an n-dimensional (pseudo-)Riemannian space, with local coordinates $x^1, \ldots, x^n$ and metric g_{ij}, the Laplace–Beltrami operator is defined by

$$L_b f = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right), \tag{1}$$

where g is the determinant of the matrix g_{ij}. When the space is either flat or constant curvature, it possesses the maximal group of isometries, which is of dimension $\frac{1}{2}n(n+1)$. The infinitesimal

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generators (Killing vectors) are just first-order differential operators which commute with the Laplace–Beltrami operator (1). This is always the case, whenever the space has isometries, but in the case of flat and constant curvature spaces, L_{b} is actually the second-order *Casimir* function of the symmetry algebra (see [6]). Since we are not going to be involved in tensorial calculations, there is no need to use upper and lower index notation. All coordinates will carry lower indices in what follows.

In this paper, we restrict attention to the case $n = 2$, but the approach is easily applied to higher dimensions. Constant curvature metrics of a given scalar curvature are related through coordinate transformations. In Section 2, we introduce a specific form of metric and its symmetry algebra (Killing vectors). We then give the form of various separation coordinate systems, which will be used later in the paper.

We use *first-order* Darboux transformations to add a *potential functions* to this differential operator in such a way that it still possesses a number of first integrals. We are therefore led to consider the operator defined by

$$Lf = L_{\mathrm{b}} f + uf, \tag{2}$$

where u is a function of the coordinates. Such Darboux transformations are related to the symmetries of the Laplace–Beltrami operator and lead to the separability of the operator L. The resulting Darboux transformations are closely related to those of the one-dimensional Schrödinger operator and the well known KdV Bäcklund chains. These give rise to very specific potential functions for which eigenfunctions can be *explicitly* calculated. It should be recalled that in the *general* separable case, the construction of eigenfunctions is reduced to the solution of a number of first- or second-order *ordinary differential equations*, which need not be *explicitly solvable*. This is analogous to the *classical* case of "integration up to quadrature". However, in the quantum case the problem will be reduced to one or more one-dimensional Schrödinger equations, which *still* need to be solved.

Eigenfunctions of the Laplace–Beltrami operator are constructed as highest weight representations of the symmetry algebra and these are used to build eigenfunctions of the Darboux related operators. These operators are shown to possess a commuting operator, which then acts invariantly on the eigenspaces of L.

For special potentials there exist *more* than n (independent) commuting operators (only n of which are in involution). Such systems are called *super-integrable*. We consider these in Section 6.

2. The metric and its symmetries

The coefficients of leading order terms in the Laplace–Beltrami operator are the coefficients of the *inverse metric* g^{ij}. We consider the constant curvature metric with inverse:

$$g^{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 - y \end{pmatrix}. \tag{3}$$

Remark 2.1. The choice of coordinates here is motivated by the relationship to Krall–Sheffer operators [9], but otherwise quite arbitrary. We later introduce a variety of separation coordinates, which will actually be used in most calculations.

A convenient basis of Killing vectors is

$$\mathbf{H} = 4x\partial_x, \quad \mathbf{E} = 2\sqrt{xy}\partial_y, \quad \mathbf{F} = 4\sqrt{xy}\partial_x + 2(y-1)\sqrt{\frac{y}{x}}\partial_y, \tag{4}$$

satisfying the standard commutation relations of $sl(2, \mathbf{C})$:

$$[\mathbf{H}, \mathbf{E}] = 2\mathbf{E}, \quad [\mathbf{H}, \mathbf{F}] = -2\mathbf{F}, \quad [\mathbf{E}, \mathbf{F}] = \mathbf{H}. \tag{5}$$

The Laplace–Beltrami operator for the metric (3) is proportional to the quadratic Casimir operator:

$$L_b = \frac{1}{16}(\mathbf{H}^2 + 2\mathbf{EF} + 2\mathbf{FE}) = x^2\partial_x^2 + 2xy\partial_x\partial_y + (y^2 - y)\partial_y^2 + \frac{3}{2}x\partial_x + \frac{1}{2}(3y-1)\partial_y. \tag{6}$$

Since the Killing vectors commute with L_b, they act on its eigenfunctions without changing eigenvalue, corresponding to degeneracy. We can therefore use the representation theory of $sl(2, \mathbf{C})$ to build eigenfunctions of L_b. In general, the symmetry algebra of L_b plays no role when a potential is added. An exceptional case is when the potential is an invariant function of one of the geometric symmetries of L_b, but this is rather trivial. Later we see that symmetries play a role in Darboux transformations, which enable us to build special potentials whose eigenfunctions can be built out of those of the "bare operator" L_b. However, we first consider the construction of second-order operators and separable coordinates.

Suppose the operator

$$I_1 = K^{11}\partial_x^2 + 2K^{12}\partial_x\partial_y + K^{22}\partial_y^2 + k^1\partial_x + k^2\partial_y + v(x, y)$$

commutes with the operator (2). Then K^{ij} are components of a rank two Killing tensor (see [3]). For *flat* and *constant curvature* metrics these are built out of Killing vectors by symmetric tensor products [2,8]. The function $v(x, y)$ then satisfies a pair of first-order partial differential equations, whose integrability conditions give rise to a second-order partial differential equation for $u(x, y)$. Writing this operator in characteristic coordinates leads to separation of variables. We give three examples below in order to collect formulae for later sections. There is a considerable amount of freedom in choosing separation coordinates. We choose coordinates which render the metric in diagonal, conformally flat form.

Remark 2.2. In two dimensions *all* metrics are conformally flat. However, in *all* dimensions, spaces of constant curvature are automatically conformally flat. The construction used here is specifically tailored to this case and can therefore be used in higher dimensional examples.

If we are just interested in separable systems, we could take *any* quadratic expression in Killing vectors. However, we find that as a consequence of building a potential function by the application of a sequence of first-order Darboux transformations, the resulting operator L commutes with a second-order operator of specific form

$$I_1 = \mathbf{K}^2 + v(x, y),$$

where $\mathbf{K}$ is a Killing vector. For this reason, the examples below are of this form.

Example 2.1 (The Killing tensor $\mathbf{H}^2$). Suppose

$$I_1 = \mathbf{H}^2 + v(x, y) = x^2\partial_x^2 + x\partial_x + v(x, y).$$

If I_1 had been a general second-order differential operator, the expansion of $[L, I_1] = 0$ would have contained terms of all orders from three to zero. Since $\mathbf{H}^2$ is a Killing tensor, the third- and second-order terms automatically vanish. The first-order terms give us

$$v_x = (1-y)u_x, \quad v_y = xu_x \quad \Rightarrow \quad \partial_x(xu_x + (y-1)u_y + u) = 0, \tag{7}$$

with general solution

$$u(x,y) = \frac{u_1\left(\frac{y-1}{x}\right)}{y-1} + u_2(y), \quad v(x,y) = -u_1\left(\frac{y-1}{x}\right).$$

We see the characteristic property of a separable system in two dimensions, with the potential depending upon two arbitrary functions of a single variable. These (or *any* function of them) are the separation variables for the operator (2) with this potential, which is itself a consequence of our choice of second-order Killing tensor. We choose functions

$$q_1 = Q_1\left(\frac{y-1}{x}\right), \qquad q_2 = Q_2(y)$$

such that

$$L_{\mathrm{b}} Q_i = 0 \Rightarrow \begin{cases} \frac{x}{y-1} Q_1'' + Q_1' = 0, \\ y(y-1)Q_2'' + \frac{1}{2}(3y-1)Q_2' = 0. \end{cases}$$

Remark 2.3. This condition generally holds in two dimensions for metrics in "Riemannian" form (explicitly conformally flat). In higher dimensions a slightly more complicated condition holds.

This gives the coordinates

$$q_1 = \frac{1}{4}\ln\left(\frac{x}{y-1}\right), \qquad q_2 = \frac{1}{4}\ln\left(\frac{\sqrt{y}+1}{\sqrt{y}-1}\right),$$

$$\text{so} \quad x = \frac{e^{4q_1}}{\sinh^2(2q_2)}, \quad y = \coth^2(2q_2),$$

after which (relabelling u_i)

$$L = -\frac{1}{16}\sinh^2(2q_2)(\partial_1^2 - \partial_2^2 + u_1(q_1) - u_2(q_2)), \qquad I_1 = \partial_1^2 + u_1(q_1) \tag{8}$$

and

$$\mathbf{H} = \partial_1, \qquad \mathbf{E} = -\frac{1}{2}e^{2q_1}(\cosh(2q_2)\partial_1 + \sinh(2q_2)\partial_2),$$

$$\mathbf{F} = \frac{1}{2}e^{-2q_1}(\cosh(2q_2)\partial_1 - \sinh(2q_2)\partial_2).$$

The fact that q_1 "straightens" $\mathbf{H}$ is a consequence of our choice of Killing tensor.

Example 2.2 (The Killing tensor $\mathbf{E}^2$). Suppose

$$I_1 = \mathbf{E}^2 + v(x,y) = 4xy\partial_y^2 + 2x\partial_y + v(x,y).$$

Once again, the expansion of $[L, I_1] = 0$ contains only first- and zeroth-order terms giving

$$v_x = 4yu_y, \quad v_y = -4xu_y \quad \Rightarrow \quad \partial_y(xu_x + yu_y + u) = 0, \tag{9}$$

with general solution

$$u(x,y)=u_1(x)+\frac{u_2(\frac{y}{x})}{x},\qquad v(x,y)=-4u_2\left(\frac{y}{x}\right).$$

Again the potential depends upon two arbitrary functions of a single variable. This time we choose the coordinates

$$q_1=\frac{1}{\sqrt{x}},\qquad q_2=\sqrt{\frac{y}{x}},\quad \text{so } x=\frac{1}{q_1^2},\ y=\frac{q_2^2}{q_1^2},$$

after which (relabelling u_i)

$$L=\frac{1}{4}q_1^2(\partial_1^2-\partial_2^2+u_1(q_1)-u_2(q_2)),\qquad I_1=\partial_2^2+u_2(q_2) \tag{10}$$

and

$$\mathbf{H}=-2(q_1\partial_1+q_2\partial_2),\qquad \mathbf{E}=\partial_2,\qquad \mathbf{F}=-2q_1q_2\partial_1-(q_1^2+q_2^2)\partial_2.$$

This time $\mathbf{E}$ is "straightened".

Example 2.3 (The Killing tensor $\mathbf{F}^2$). Suppose

$$I_1=\mathbf{F}^2+v(x,y)=16xy\partial_x^2+16y(y-1)\partial_x\partial_y$$
$$+4\frac{y(y-1)^2}{x}\partial_y^2+4(3y-1)\partial_x+2\frac{(y-1)^2}{x}\partial_y+v(x,y).$$

Once again, the expansion of $[L,I_1]=0$ contains only first- and zeroth-order terms giving

$$v_x=\frac{4y(1-y)}{x^2}(2xu_x+(y-1)u_y),\qquad v_y=\frac{4(1+y)}{x}(2xu_x+(y-1)u_y),$$

with integrability conditions

$$x^2(1+y)u_{xx}+\frac{1}{2}x(y-1)(3y+1)u_{xy}+\frac{1}{2}y(y-1)^2u_{yy}$$
$$+x(2y-1)u_x+(y-1)^2u_y=0. \tag{11}$$

This is a hyperbolic equations with characteristic coordinates

$$q_1=\frac{\sqrt{x}}{y-1},\qquad q_2=\frac{\sqrt{xy}}{y-1},\qquad \text{so } x=\frac{(q_2^2-q_1^2)^2}{q_1^2},\ y=\frac{q_2^2}{q_1^2}.$$

These were once again chosen to render $L_\flat$ in "Riemannian" form. In these coordinates, (11) simplifies to

$$q_1u_{q_1q_2}-2u_{q_2}=0\quad\Rightarrow\quad u(q_1,q_2)=\frac{1}{4}q_1^2(u_1(q_1)-u_2(q_2)).$$

The operators L and I_1 now take the form:

$$L=\frac{1}{4}q_1^2(\partial_1^2-\partial_2^2+u_1(q_1)-u_2(q_2)),\qquad I_1=\partial_2^2+u_2(q_2) \tag{12}$$

with

$$\mathbf{H}=2(q_1\partial_1+q_2\partial_2),\qquad \mathbf{F}=\partial_2,\qquad \mathbf{E}=-2q_1q_2\partial_1-(q_1^2+q_2^2)\partial_2.$$

This time $\mathbf{F}$ is "straightened".

Involution. Since the changes of coordinates in Examples 2.2 and 2.3 lead to *exactly the same* form of operator (same *metric*), we have the following transformations:

$$\bar{q}_1 = \frac{q_1}{q_2^2 - q_1^2}, \qquad \bar{q}_2 = \frac{q_2}{q_2^2 - q_1^2},$$

$$\text{with inverse} \quad q_1 = \frac{\bar{q}_1}{\bar{q}_2^2 - \bar{q}_1^2}, \qquad q_2 = \frac{\bar{q}_2}{\bar{q}_2^2 - \bar{q}_1^2}, \tag{13}$$

where we have used (q_1, q_2) and $(\bar{q}_1, \bar{q}_2)$ to represent the two different cases (related to a *single point* (x, y)). Since the transformation and its inverse have the same form, this is an *involution*, which, in fact, gives a concrete realisation of the Lie algebra automorphism:

$$\mathbf{E} \leftrightarrow \mathbf{F}, \qquad \mathbf{H} \to -\mathbf{H}.$$

On the other hand, thinking of the separated metric as the "fixed object", we have an involution which realises the same automorphism in the (x, y) coordinates:

$$\bar{x} = \frac{(y-1)^2}{x}, \qquad \bar{y} = y. \tag{14}$$

3. Eigenfunctions of the Laplace–Beltrami operator

We use the *highest weight representations* of the symmetry algebra $sl(2, \mathbf{C})$ to construct eigenfunctions of L_{b}. Since $\mathbf{H}$ and L_{b} commute, they share eigenfunctions and for $\mathbf{H}$ these are built by the highest weight construction. Furthermore, starting with *any* eigenfunction of L_{b}, we may use the symmetry algebra to construct further eigenfunctions (with the same eigenvalue). Since this eigenspace is *invariant* under the action of $sl(2, \mathbf{C})$ (by construction), it can be decomposed into irreducible components, which are just weight spaces. Therefore, *all* eigenfunctions of L_{b} can be written as linear combinations of those we construct below.

A *highest weight vector* ψ_1^m, of weight $2m$, satisfies

$$\mathbf{E}\psi_1^m = 0, \qquad \mathbf{H}\psi_1^m = 2m\psi_1^m,$$

which constitute a pair of partial differential equations for the eigenfunction. These are compatible on the zeros of the differential operator $\mathbf{E}$, since

$$\mathbf{HE}\psi_1^m - \mathbf{EH}\psi_1^m = 2\mathbf{E}\psi_1^m = 0.$$

The specific form of ψ_1^m depends upon m and upon the choice of representation for $sl(2, \mathbf{C})$. However, the general structure of the representation is *independent* of this specific form, being a consequence only of the commutation relations (5) (see [7]).

Defining $\psi_n^m = \mathbf{F}^{n-1}\psi_1^m$, the commutation relations imply:

$$\mathbf{H}\psi_n^m = 2(m+1-n)\psi_n^m, \qquad \mathbf{E}\psi_n^m = (n-1)(2m+2-n)\psi_{n-1}^m. \tag{15}$$

Our definition of Laplace–Beltrami operator L_{b} as Casimir operator (6) implies that

$$L_{\mathrm{b}}\psi_n^m = \frac{1}{4}m(m+1)\psi_n^m, \quad \text{for all } m, n.$$

We can also construct a three-point recursion relation between these eigenfunctions, but this *is* explicitly dependent upon the representation. Let

$$\mathbf{H} = h_1\partial_{z_1} + h_2\partial_{z_2}, \qquad \mathbf{E} = e_1\partial_{z_1} + e_2\partial_{z_2}, \qquad \mathbf{F} = f_1\partial_{z_1} + f_2\partial_{z_2},$$

where h_i, etc are functions of the coordinates z_i. We have (with μ_n and a_n defined by (15))

$$\left.\begin{aligned}\mathbf{H}\psi_n^m &= \mu_n\psi_n^m,\\ \mathbf{E}\psi_n^m &= a_n\psi_{n-1}^m\end{aligned}\right\} \Rightarrow \begin{pmatrix}\partial_{z_1}\psi_n^m\\ \partial_{z_2}\psi_n^m\end{pmatrix} = \frac{1}{h_1e_2-e_1h_2}\begin{pmatrix}e_2 & -h_2\\ -e_1 & h_1\end{pmatrix}\begin{pmatrix}\mu_n\psi_n^m\\ a_n\psi_{n-1}^m\end{pmatrix}. \quad (16)$$

The relation $\psi_{n+1}^m = \mathbf{F}\psi_n^m$ then implies that

$$\psi_{n+1}^m = \frac{f_1e_2-e_1f_2}{h_1e_2-e_1h_2}\mu_n\psi_n^m + \frac{h_1f_2-f_1h_2}{h_1e_2-e_1h_2}a_n\psi_{n-1}^m. \quad (17)$$

When this is singular the representation reduces to one dimension.

For the three examples presented in Section 2, we give the explicit formulae.

The Killing tensor $\mathbf{H}^2$. Here

$$\begin{aligned}\psi_1^m &= \left(\frac{e^{2q_1}}{\sinh 2q_2}\right)^m,\\ \psi_{n+1}^m &= 2(m+1-n)e^{-2q_1}\cosh 2q_2\psi_n^m + (n-1)(2m+2-n)e^{-4q_1}\psi_{n-1}^m. \end{aligned} \quad (18)$$

The Killing tensor $\mathbf{E}^2$. Here

$$\begin{aligned}\psi_1^m &= q_1^{-m},\\ \psi_{n+1}^m &= 2(m+1-n)q_2\psi_n^m + (n-1)(2m+2-n)(q_2^2-q_1^2)\psi_{n-1}^m. \end{aligned} \quad (19)$$

The Killing tensor $\mathbf{F}^2$. Here

$$\begin{aligned}\psi_1^m &= \left(\frac{q_1^2-q_2^2}{q_1}\right)^m,\\ \psi_{n+1}^m &= 2(m+1-n)\frac{q_2}{q_2^2-q_1^2}\psi_n^m + \frac{(n-1)(2m+2-n)}{q_2^2-q_1^2}\psi_{n-1}^m. \end{aligned} \quad (20)$$

The cases of $\mathbf{E}^2$ and $\mathbf{F}^2$ are related through the involution (13). For instance,

$$\psi_1^m = q_1^{-m} \leftrightarrow \psi_1^m = \left(\frac{q_1^2-q_2^2}{q_1}\right)^m$$

under the involution.

When m is an integer, these representations are of finite dimension $2m+1$ and irreducible, but infinite dimensional otherwise.

4. Darboux transformations

Darboux transformations are analogous to *similarity transformations* in matrix theory. We require an intertwining operator between two operators of type (2), with the same L_b but different potentials:

$$L_i = L_b + u_i, \quad i = 1, 2, \qquad L_2\mathcal{D} = \mathcal{D}L_1 + \delta\mathcal{D}, \quad (21)$$

where $\mathcal{D}$ is a differential operator (if $\mathcal{D}$ is just a function, this is a "gauge transformation"). When $\delta = 0$, (21) is *isospectral*, but otherwise

$$L_1\psi = \lambda\psi \Rightarrow L_2(\mathcal{D}\psi) = (\lambda + \delta)(\mathcal{D}\psi). \tag{22}$$

The simplest case is when $\mathcal{D}$ is a first-order differential operator:

$$\mathcal{D} = a(x, y)\partial_x + b(x, y)\partial_y + w(x, y)$$

with which the leading order term of (21) is the *second order* operator

$$[L_{\rm b}, a(x, y)\partial_x + b(x, y)\partial_y],$$

which should therefore vanish. This is the condition that $a(x, y)\partial_x + b(x, y)\partial_y$ is a *Killing vector field* of the metric. We therefore choose the differential part of $\mathcal{D}$ to be any linear combination of the basis elements $\mathbf{H}$, $\mathbf{E}$, $\mathbf{F}$.

We consider operators $L_n = L_{\rm b} + u_n(q_1, q_2)$, where, the suffix n refers to the sequence of operators created by a succession of Darboux transformations:

$$L_{n+1}\mathcal{D}_n = \mathcal{D}_n L_n + \delta_n\mathcal{D}_n, \quad \mathcal{D}_n = \mathbf{K} + w_n, \tag{23}$$

where $\mathbf{K}$ is the chosen Killing vector. This gives rise to a *Bäcklund chain* which is closely related to that of the KdV equation.

The *first integral* I_1 (presented for particular examples in Section 2) is then just the Darboux transformations of the corresponding *quadratic Killing tensor* $\mathbf{K}^2$:

$$(\mathbf{K}^2 + v_{n+1})\mathcal{D}_n = \mathcal{D}_n(\mathbf{K}^2 + v_n). \tag{24}$$

The function v_n is a separated part of the potential u_n (see examples below). The equations on w_n and u_n, implied by this, are the same as those implied by the Darboux transformation (23).

We can use any coordinate system for our calculations, but adapting coordinates to the particular Killing field is convenient. This means choosing one coordinate q_i, such that $a(x, y)\partial_x + b(x, y)\partial_y = \partial_i$, after which we find that $w(x, y) = \bar{w}(q_i)$. In the examples below, we use the coordinates introduced in Section 2, but sometimes transform back to the original $x - y$ coordinates of the metric (3).

4.1. *The operator (8) with Killing tensor $\boldsymbol{H}^2$*

We consider operators

$$L_n = -\frac{1}{16}\sinh^2(2q_2)(\partial_1^2 - \partial_2^2 + u_n(q_1, q_2)),$$

where for this calculation, we have not yet separated the potential. As above, the suffix n refers to the sequence of operators created by a succession of Darboux transformations:

$$L_{n+1}\mathcal{D}_n = \mathcal{D}_n L_n + \delta_n\mathcal{D}_n, \quad \mathcal{D}_n = \partial_1 + w_n(q_1), \tag{25}$$

leading to

$$u_{n+1} = u_n - 2w_n'(q_1) - \frac{16\delta_n}{\sinh^2(2q_2)}, \quad u_n = w_n'(q_1) - w_n^2(q_1) - \sigma_n(q_2).$$

These give two formulae for u_{n+1}, which can be equated, leading to the following Bäcklund chain:

$$w_{n+1}' + w_n' + w_n^2 - w_{n+1}^2 - \sigma_{n+1} + \sigma_n + \frac{16\delta_n}{\sinh^2(2q_2)} = 0,$$

which separates into the pair of chains

$$w'_{n+1} + w'_n + w_n^2 - w_{n+1}^2 = \gamma_n, \quad \sigma_{n+1} - \sigma_n - \frac{16\delta_n}{\sinh^2(2q_2)} = \gamma_n. \tag{26}$$

The functions $w_n(q_1)$ satisfy the standard Bäcklund chain of the one-dimensional Schrödinger operator of KdV theory [10,11], so the solutions are well known. For a given solution of w_n, u_n, the transformation of Example 2.1 gives a potential in terms of x, y.

"Soliton" solutions. With $w_n = -(n+1)\tanh q_1$, we get

$$\gamma_n = -2n - 3, u_n = n(n+1)\,\text{sech}^2 q_1 - (n+1)^2 - \sigma_n(q_2). \tag{27}$$

With $\sigma_0 = -1$ and $\delta_n = -\frac{1}{8}(n+1)$ we have

$$\sigma_n = -\frac{n(n+1)}{\sinh^2 2q_2} - (n+1)^2 \quad \text{and} \quad u_n = n(n+1)\,\text{sech}^2 q_1 + \frac{n(n+1)}{\sinh^2 2q_2}.$$

Rational solutions. With $w_n = -\frac{n+1}{q_1}$, we get

$$\gamma_n = 0, \qquad u_n = -\frac{n(n+1)}{q_1^2} - \sigma_n(q_2). \tag{28}$$

With $\sigma_0 = 0$ and $\delta_n = -\frac{1}{8}(n+1)$ we have

$$\sigma_n = -\frac{n(n+1)}{\sinh^2 2q_2} \quad \text{and} \quad u_n = \frac{n(n+1)}{\sinh^2 2q_2} - \frac{n(n+1)}{q_1^2}.$$

The first integral I_1. Under the Darboux transformation the first integral satisfies the "usual" KdV Darboux transformation

$$(\partial_1^2 + u_{n+1,1}(q_1))(\partial_1 + w_n(q_1)) = (\partial_1 + w_n(q_1))(\partial_1^2 + u_{n,1}(q_1)),$$

where $\partial_1 = \mathbf{H}$, with $u_{n,1}(q_1)$ denoting the separated q_1-dependent part of the potential $u_n(q_1, q_2)$.

4.2. The operators (10) and (12) with Killing tensors $\boldsymbol{E}^2$ and $\boldsymbol{F}^2$

We can treat Examples 2.2 and 2.3 together, since in their respective separation variables they lead to the same operator

$$L_n = \frac{1}{4} q_1^2(\partial_1^2 - \partial_2^2 - u_n(q_1, q_2)),$$

where again, we have not yet separated the potential. These are related through a succession of Darboux transformations:

$$L_{n+1}\mathcal{D}_n = \mathcal{D}_n L_n + \delta_n \mathcal{D}_n, \quad \mathcal{D}_n = \partial_2 + w_n(q_2), \tag{29}$$

leading to

$$u_{n+1} = u_n - 2w'_n(q_2) - \frac{4\delta_n}{q_1^2}, \qquad u_n = w'_n(q_2) - w_n^2(q_2) - \sigma_n(q_1).$$

As before, we obtain the following Bäcklund chain:

$$w'_{n+1} + w'_n + w_n^2 - w_{n+1}^2 - \sigma_{n+1} + \sigma_n + \frac{4\delta_n}{q_1^2} = 0,$$

which separates into the pair of chains

$$w'_{n+1}+w'_n+w_n^2-w_{n+1}^2=\gamma_n,\quad \sigma_{n+1}-\sigma_n-\frac{4\delta_n}{q_1^2}=\gamma_n. \tag{30}$$

The functions $w_n(q_2)$ satisfy the *same* Bäcklund chain, with the only difference being in the relation satisfied by $\sigma_n(q_1)$. For a given solution of w_n, u_n, the respective transformations of Examples 2.2 and 2.3 give (generally) two *different* potentials in terms of x, y.

"Soliton" solutions. With $w_n=-(n+1)\tanh q_2$, we get

$$\gamma_n=-2n-3,\quad u_n=n(n+1)\operatorname{sech}^2 q_2-(n+1)^2-\sigma_n(q_1). \tag{31}$$

With $\sigma_0=-1$ and $\delta_n=-\frac{1}{2}(n+1)$ we have

$$\sigma_n=-\frac{n(n+1)}{q_1^2}-(n+1)^2 \quad\text{and}\quad u_n=n(n+1)\operatorname{sech}^2 q_2+\frac{n(n+1)}{q_1^2}.$$

Rational solutions. With $w_n=-\frac{n+1}{q_2}$, we get

$$\gamma_n=0,\quad u_n=-\frac{n(n+1)}{q_2^2}-\sigma_n(q_1). \tag{32}$$

With $\sigma_0=0$ and $\delta_n=-\frac{1}{2}(n+1)$ we have

$$\sigma_n=-\frac{n(n+1)}{q_1^2}\quad\text{and}\quad u_n=\frac{n(n+1)}{q_1^2}-\frac{n(n+1)}{q_2^2}.$$

Adler–Moser rational solutions. With $w_n=-\partial_{q_2}\log\varphi_n(q_2)$, we get $u_n=-\frac{\varphi_n''}{\varphi_n}-\sigma_n(q_1)$. The further substitution

$$\varphi_n=\frac{P_n(q_2)}{P_{n-1}(q_2)}\Rightarrow P_{n-1}P'_{n+1}-P'_{n-1}P_{n+1}=(2n+3)P_n^2.$$

It can be shown [1] that the functions P_n, recursively defined by this formula (with $P_{-1}=1$, $P_0=q_2$) are *polynomials* (just the "τ-functions" for the rational solutions of the KdV equation). The degree of P_n is $\frac{1}{2}(n+1)(n+2)$. Some of the lower members of this list are given by

$$\begin{aligned}
&P_0=q_2,\quad P_1=q_2^3+t_1,\quad P_2=q_2^6+5t_1q_2^3-5t_1^2,\\
&w_0=-\frac{1}{q_2},\ w_1=\frac{t_1-2q_2^3}{q_2(q_2^3+t_1)},\quad w_2=-3\frac{q_2^8+2t_1q_2^5+10t_1^2q_2^2}{q_2^9+6t_1q_2^6-5t_1^3},\\
&u_0=0,\quad u_1=-\frac{2}{q_2^2}-\sigma_1(q_1),\ u_2=-6\frac{q_2^4-2t_1q_2}{(q_2^3+t_1)^2}-\sigma_2(q_1)
\end{aligned} \tag{33}$$

where once again we choose $\sigma_n=-\frac{n(n+1)}{q_1^2}$. When $t_1=0$, these formulae reduce to the previous case of "rational solutions".

The first integral I_1. Under the Darboux transformation the first integral satisfies the "usual" KdV Darboux transformation

$$(\partial_2^2+u_{n+1,2}(q_2))(\partial_2+w_n(q_2))=(\partial_2+w_n(q_2))(\partial_2^2+u_{n,2}(q_2)),$$

where $\partial_2=\mathbf{E}$ or $\partial_2=\mathbf{F}$, depending upon the case and where $u_{n,2}(q_2)$ denotes the separated q_2-dependent part of the potential $u_n(q_1,q_2)$.

Remark 4.1. With $t_1 = -12t$, the Adler–Moser potential satisfies the KdV equation

$$Q(q_2, t) = u_{2,2} = -6\frac{q_2(q_2^3 + 24t)}{(q_2^3 - 12t)^2} \Rightarrow Q_t = Q_{q_2q_2q_2} + 6QQ_{q_2}.$$

The operator $I_1 = \partial_2^2 + Q(q_2, t)$ is the usual Lax operator, satisfying

$$I_{1t} = [P, I_1],$$

where

$$P = 4\partial_2^3 + 6Q\partial_2 + 3Q_{q_2},$$

but we may also use the above L, which also satisfies $L_t = [P, L]$.

5. Eigenfunctions of Darboux related operators

We have seen how to construct the eigenfunctions of the Laplace–Beltrami operator L_b through the representation theory of the symmetry algebra (in this case $sl(2, \mathbf{C})$). Formula (22) shows that a Darboux transformation carries forward these eigenfunctions, as well as building the potential functions. For each m we start with ψ_1^m and build ψ_i^m with the recursion relation (17) (or just by acting upon ψ_1^m with $\mathbf{F}$) to build the eigenfunctions of L_b. We then act on these with $\mathcal{D}_i$, $i = 0, 1, \ldots,$ to construct the eigenfunctions of $L_1, L_2, \ldots,$ defined by

$$\psi_{i,n}^m = \mathcal{D}_{n-1} \cdots \mathcal{D}_0 \psi_i^m \quad \text{with} \quad L_n \psi_{i,n}^m = \lambda_{m,n} \psi_{i,n}^m, \quad \lambda_{m,n} = \frac{1}{4}m(m+1) + \sum_{i=0}^{n-1} \delta_i. \tag{34}$$

For each n, eigenspaces of L_n are *invariant* under the action of the first integral $I_1^{(n)}$. Depending upon the example, $I_1^{(n)}$ may act as a "ladder operator" or may act diagonally on the sequence $\psi_{i,n}^m$. We may consider this as a "ghost" of the action of part of the symmetry algebra of L_b, which generally do not act invariantly on these eigenfunctions.

5.1. *The case $\mathcal{D} = \boldsymbol{H} + w$*

Since $\mathbf{K} = \mathbf{H}$, $I_1^{(n)}$ is Darboux related to $\mathbf{H}^2$ through the *isospectral* relation (24). Since ψ_i^m satisfies

$$\mathbf{H}^2 \psi_i^m = \mu_i^2 \psi_i^m,$$

with $\mu_i = 2(m + 1 - i)$, the eigenfunctions $\psi_{i,n}^m$ satisfy

$$L_n \psi_{i,n}^m = \lambda_{m,n} \psi_{i,n}^m \quad \text{and} \quad I_1^{(n)} \psi_{i,n}^m = \mu_i^2 \psi_{i,n}^m.$$

The first two eigenfunctions of L_b are

$$\psi_1^m = \frac{e^{2mq_1}}{(\sinh 2q_2)^m}, \qquad \psi_2^m = 2me^{2(m-1)q_1} \frac{\cosh 2q_2}{(\sinh 2q_2)^m}. \tag{35}$$

We consider two examples of Darboux transformation.

The “Soliton” potentials. With $w_n = -(n+1)\tanh q_1$, we obtain the “soliton” potentials (27). Using the Darboux transformation $\mathcal{D}_0$, we find

$$\psi^m_{1,1} = e^{2mq_1}\frac{(2m-\tanh q_1)}{(\sinh 2q_2)^m}, \qquad \psi^m_{2,1} = 2me^{2(m-1)q_1}\frac{(2(m-1)-\tanh q_1)\cosh 2q_2}{(\sinh 2q_2)^m}.$$

Now using the Darboux transformation $\mathcal{D}_1$, we find

$$\psi^m_{1,2} = e^{2mq_1}\frac{(4m^2-1-6m\,\tanh q_1+3\,\tanh^2 q_1)}{(\sinh 2q_2)^m},$$

$$\psi^m_{2,2} = 2me^{2(m-1)q_1}\frac{(4(m-1)^2-1-6(m-1)\,\tanh q_1+3\,\tanh^2 q_1)\cosh 2q_2}{(\sinh 2q_2)^m}.$$

These eigenfunctions are being generated *algebraically*, with no control over analytic or asymptotic properties. However, the well known bound states of $I_1^{(n)}$, for each n, are included.

When $m = \frac{1}{2}$, the eigenfunction $\psi^m_{1,1}$ is the bound state for $I_1^{(1)} = \partial_1^2 + 2\text{sech}^2\, q_1$, with eigenvalue $\mu_1^2 = 1$.

When $m = \frac{1}{2}$, the eigenfunction

$$\psi^m_{1,2} = \frac{-3}{\sqrt{\sinh 2q_2}}\,\tanh q_1\,\text{sech}\, q_1$$

is a bound state for $I_1^{(2)} = \partial_1^2 + 6\,\text{sech}^2 q_1$, with eigenvalue $\mu_1^2 = 1$. When $m = 1$, the eigenfunction

$$\psi^m_{1,2} = \frac{3}{\sinh 2q_2}\,\text{sech}^2\, q_1$$

is the second bound state for $I_1^{(2)}$, with eigenvalue $\mu_1^2 = 4$.

The “Rational” potentials. With $w_n = -\frac{(n+1)}{q_1}$, we obtain the “rational” potentials (28). We have the same eigenfunctions (35) of $L_{\rm b}$, but now the form of $\mathcal{D}_n$ is different. $\mathcal{D}_0$ now generates

$$\psi^m_{1,1} = \left(\frac{2mq_1-1}{q_1}\right)e^{2mq_1}\frac{1}{(\sinh 2q_2)^m},$$

$$\psi^m_{2,1} = 2m\left(\frac{2(m-1)q_1-1}{q_1}\right)e^{2(m-1)q_1}\frac{\cosh 2q_2}{(\sinh 2q_2)^m}.$$

Now using the Darboux transformation $\mathcal{D}_1$, we find

$$\psi^m_{1,2} = \left(\frac{4m^2q_1^2-6mq_1+3}{q_1^2}\right)e^{2mq_1}\frac{1}{(\sinh 2q_2)^m},$$

$$\psi^m_{2,2} = 2m\left(\frac{4(m-1)^2q_1^2-6(m-1)q_1+3}{q_1^2}\right)e^{2(m-1)q_1}\frac{\cosh 2q_2}{(\sinh 2q_2)^m}.$$

5.2. The case $\mathcal{D} = \boldsymbol{F} + w$

Since $\mathbf{K} = \mathbf{F}$,

$$I_1^{(n)}\mathcal{D} = \mathcal{D}\mathbf{F}^2, \quad \text{with} \quad \mathcal{D} = \mathcal{D}_{n-1}\cdots\mathcal{D}_0,$$

so

$$\mathbf{F}^2\psi_i^m = \psi_{i+2}^m \Rightarrow I_1^{(n)}\psi_{i,n}^m = \psi_{i+2,n}^m, \quad \text{for each} \quad n.$$

The first three eigenfunctions of L_b are

$$\psi_1^m = \frac{(q_1^2 - q_2^2)^m}{q_1^m}, \qquad \psi_2^m = \frac{-2mq_2(q_1^2 - q_2^2)^{m-1}}{q_1^m},$$

$$\psi_3^m = \frac{2m((2m-1)q_2^2 - q_1^2)(q_1^2 - q_2^2)^{m-2}}{q_1^m}.$$

We consider two examples of Darboux transformation.

The "rational" potentials. With $w_n = -\frac{(n+1)}{q_2}$, we obtain the "rational" potentials (32). The commuting operator $I_1^{(n)}$ takes the form

$$I_1^{(n)} = \partial_2^2 - \frac{n(n+1)}{q_2^2}.$$

The Darboux transformation $\mathcal{D}_0$ generates

$$\psi_{1,1}^m = -\frac{(2m-1)q_2^2 + q_1^2}{q_2}\left(\frac{(q_1^2 - q_2^2)^{m-1}}{q_1^m}\right),$$

$$\psi_{2,1}^m = 4m(m-1)q_2^2\left(\frac{(q_1^2 - q_2^2)^{m-2}}{q_1^m}\right). \tag{36}$$

Now using the Darboux transformation $\mathcal{D}_1$, we find

$$\psi_{1,2}^m = \frac{(2m-1)(2m-3)q_2^4 + 2(2m-3)q_1^2q_2^2 + 3q_1^4}{q_2^2}\left(\frac{(q_1^2 - q_2^2)^{m-2}}{q_1^m}\right),$$

$$\psi_{2,2}^m = -8m(m-1)(m-2)q_2^3\left(\frac{(q_1^2 - q_2^2)^{m-3}}{q_1^m}\right). \tag{37}$$

For integer values of m, the representation of $sl(2, \mathbf{C})$ is of dimension $2m + 1$. In the present case, this is easily seen, since ψ_1^m is a polynomial of degree $2m$ in q_2. Since $\mathbf{F} = \partial_2$, this just reduces the degree by 1, so ψ_{2m+1}^m is of degree zero and thus in the kernel of $\mathbf{F}$. For integer values of m, the maximum number of nontrivial eigenfunctions $\psi_{i,n}^m$ is therefore $2m + 1$. However, it can be seen in the above formulae that some members of the sequence can be rendered *zero* by the Darboux transformation. In fact, since

$$\mathcal{D}\, q_2^N = (N - 2n + 1), \ldots, (N-3)(N-1)q_2^{N-n},$$

we see that $\mathcal{D}q_2^N = 0$, for $N = 1, 3, \ldots, 2n-1$ (*odd* values only). Since ψ_1^m only contains terms of even degree, ψ_{2k+1}^m, $k = 0, \ldots, m$, only contain terms of *even* degree, whilst ψ_{2k}^m, $k = 1, \ldots, m$, only contain *odd* terms. More precisely, $\psi_{2(m-k)}^m$, $k = 0, \ldots, m-1$, contains odd degrees *up to* $2k + 1$, so is *first* rendered zero by $\mathcal{D}$ when $n = k + 1$. Furthermore, if $n < m$, then $\psi_{2(m-n)}^m$ is a linear combination of $q_2, q_2^3, \ldots, q_2^{2n+1}$ and only the highest term survives the Darboux transformation, with

$$\mathcal{D}q_2^{2n+1} = 2^n n! q_2^{n+1}.$$

After $\mathcal{D}$, $I_1^{(n)} = \partial_2^2 - \frac{n(n+1)}{q_2^2}$, whose kernel is spanned by $\{q_2^{n+1}, q_2^{-n}\}$, so $I_1^{(n)}\psi^m_{2(m-n),n} = 0$. The action of $I_1^{(n)}$ on eigenfunctions is therefore

$$I_1^{(n)} : \psi^m_{1,n} \to \psi^m_{3,n} \to \cdots \to \psi^m_{2m+1,n} \to 0,$$
$$I_1^{(n)} : \psi^m_{2,n} \to \psi^m_{4,n} \to \cdots \to \psi^m_{2(m-n),n} \to 0, \quad \text{for} \quad n < m.$$

When $n \geq m$, all the "even numbered" eigenfunctions are zero.

The Adler–Moser rational potentials. This is just a deformation (with parameter t_1) of the previous case. The general structure and formulae are the same, but the *explicit formulae* look very different. The first Darboux transformation and potential function are just the same (since $w_0 = -\frac{1}{q_2}$), so the eigenfunctions $\psi^m_{i,1}$ are identical to those of (36). However, the second Darboux transformation introduces the parameter t_1, which deforms the eigenfunctions (37):

$$\psi^m_{i,2} = \left(\frac{(q_1^2 - q_2^2)^{m-i-1}}{q_1^m}\right)\left(\frac{\psi^m_{i,2}(0) + t_1\psi^m_{i,2}(1)}{q_2^3 + t_1}\right).$$

For instance, for $i = 1, 2$, we have

$$\psi^m_{1,2}(0) = q_2((2m-1)(2m-3)q_2^4 + 2(2m-3)q_1^2q_2^2 + 3q_1^4),$$
$$\psi^m_{1,2}(1) = 2m\,((2m-1)q_2^2 - q_1^2), \qquad \psi^m_{2,2}(0) = -8m(m-1)(m-2)q_2^6,$$
$$\psi^m_{2,2}(1) = -4m(m-1)q_2((2m-1)q_2^2 - 3q_1^2).$$

When $t_1 = 0$, these reduce to (37).

5.3. The case $\mathcal{D} = \boldsymbol{E} + w$

There is no need to study this case in detail, since it is related through the involution (13) to the previous case, as already noted. Since

$$\psi_1^m = \left(\frac{q_1^2 - q_2^2}{q_1}\right)^m \quad \longleftrightarrow \quad \psi_1^m = q_1^{-m}$$

and since

$$\mathbf{E} \leftrightarrow \mathbf{F},$$

we obtain the same sequence of eigenfunctions for the Laplace–Beltrami operator (which is invariant under the involution). Furthermore, the Darboux transformations have the same form, so they produce the same collection of eigenfunctions for the Darboux related operators (but in reverse order).

In the x–y coordinates, the formulae *will* be different.

6. Super-integrable cases

In two dimensions it is enough to have just one operator which commutes with L, in order to proclaim *complete integrability*. For *separable potentials* the problem of calculating eigenfunctions is reduced to analysing *ordinary* differential equations, but nothing *explicit* can be calculated without further assumptions. The special potentials considered in this paper were obtained through Darboux transformations, which give us a tool for *explicitly constructing* eigenfunctions.

Another special class of potentials can be found, with the property of being separable in two different coordinate systems. These are characterised by the existence of an *additional, independent* operator, which commutes with L. The three operators will not, of course, be *in involution*. The additional commuting operator may be just a Killing vector of L_{b}, in which case the potential has a geometric symmetry. A more interesting case is when there exists an additional *second-order* commuting operator.

If we require that *two* of our chosen operators I_1 should commute with L, then the potential function must satisfy *two sets of integrability conditions*. In this case, the two arbitrary functions in our previous examples are *fixed up to a finite number of parameters*. In the x–y coordinates, the resulting potential functions are *rational*.

Example 6.1 (The Killing tensors $\mathbf{H}^2$ and $\mathbf{E}^2$). Requiring *both* $I_1 = \mathbf{H}^2 + v_1$ and $I_2 = \mathbf{E}^2 + v_2$ to commute with L means that the potential function $u(x, y)$ must satisfy *both* Eqs. (7) and (9). The resulting operators are

$$L = L_{\mathrm{b}} + \frac{c_0}{x} + \frac{c_1}{y} + c_2\,\frac{y-1}{x^2}, \qquad I_1 = \mathbf{E}^2 - 4\left(c_1\frac{x}{y} + c_2\frac{y}{x}\right),$$
$$I_2 = \mathbf{H}^2 - 16\left(c_0\left(\frac{y-1}{x}\right) + c_2\left(\frac{y-1}{x}\right)^2\right). \tag{38}$$

This L can be *gauged* to the form

$$L = x^2\partial_x^2 + 2xy\partial_x\partial_y + (y^2 - y)\partial_y^2 + (bx + e_1)\partial_x + (by + e_2)\partial_y,$$

which is one of the class of operators introduced by Krall and Sheffer [9] in the context of orthogonal polynomials in two dimensions. It was shown in [5] that the operators I_1 and I_2 can be used to build sequences of *polynomial* eigenfunctions for this gauged form of L.

Example 6.2 (The Killing tensors $\mathbf{H}^2$ and $\mathbf{F}^2$). Now the potential function $u(x, y)$ must satisfy Eqs. (7) and (11). The resulting operators are

$$L = L_{\mathrm{b}} + c_0\,\frac{x}{(y-1)^2} + \frac{c_1}{y} + c_2\,\frac{x^2}{(y-1)^3},$$
$$I_1 = \mathbf{F}^2 - 4\left(c_1\,\frac{(y-1)^2}{xy} + c_2\,\frac{xy}{(y-1)^2}\right),$$
$$I_2 = \mathbf{H}^2 - 16\left(c_0\left(\frac{x}{y-1}\right) + c_2\left(\frac{x}{y-1}\right)^2\right). \tag{39}$$

Example 6.3 (The Killing tensors $\mathbf{E}^2$ and $\mathbf{F}^2$). Now the potential function $u(x, y)$ must satisfy Eqs. (9) and (11). The resulting operators are

$$L = L_{\mathrm{b}} + \frac{c_1}{y}, \qquad I_1 = \mathbf{E}^2 - 4c_1\frac{x}{y}, \qquad I_2 = \mathbf{F}^2 - 4c_1\frac{(y-1)^2}{xy}. \tag{40}$$

Remark 6.1. Under the involution (14), cases (38) and (39) interchange, with case (40) being invariant. The operators exchange in the obvious way, reflecting the Lie algebra automorphism $\mathbf{E} \leftrightarrow \mathbf{F}, \mathbf{H} \to -\mathbf{H}$.

Not all of the above super-integrable cases are Darboux related to L_b, but some are. This is best seen in separation coordinates.

Example 6.4 (Case (38)). We write the operators (38) in the separation coordinates of (10):

$$L = \frac{1}{4}q_1^2\left(\partial_1^2 - \partial_2^2 + 4c_0 + \frac{4c_1}{q_2^2} + 4c_2(q_2^2 - q_1^2)\right), \qquad I_1 = \partial_2^2 - 4\left(\frac{c_1}{q_2^2} + c_2 q_2^2\right),$$
$$I_2 = 4(q_1\partial_1 + q_2\partial_2)^2 - 16(c_0(q_2^2 - q_1^2) + c_2(q_2^2 - q_1^2)^2). \tag{41}$$

This L can only be obtained from L_b through Darboux transformation if $c_0 = c_2 = 0$, in which case L is also of type (40). This means that, as well as the operators (41) (with $c_0 = c_2 = 0$), this L also commutes with

$$I_3 = \mathbf{F}^2 - 4c_1\frac{(q_2^2 - q_1^2)^2}{q_2^2} = (2q_1q_2\partial_1 + (q_1^2 + q_2^2)\partial_2)^2 - 4c_1\frac{(q_2^2 - q_1^2)^2}{q_2^2}.$$

Under the involution (13), L and I_2 are invariant in this reduction, with $I_1 \leftrightarrow I_3$. Since $I_2 = \mathbf{H}^2$ in this reduction, we may replace it by $\mathbf{H}$. These three operators, for *arbitrary* c_1, satisfy the commutation relations of a *polynomial extension* of $sl(2, \mathbf{C})$:

$$[\mathbf{H}, I_1^{(c_1)}] = 4I_1^{(c_1)}, \qquad [\mathbf{H}, I_3^{(c_1)}] = -4I_3^{(c_1)},$$
$$[I_1^{(c_1)}, I_3^{(c_1)}] = 2(8c_1 - 1)\mathbf{H} - \mathbf{H}^3 + 16\mathbf{H}L_{c_1}.$$

They also satisfy the relation:

$$I_1I_3 + I_3I_1 - \frac{1}{8}\mathbf{H}^4 + 4\mathbf{H}^2L - 32L^2 + \frac{1}{2}(8c_1 - 5)\mathbf{H}^2 + 16(4c_1 + 1)L = 16c_1(2c_1 - 1).$$

For the values $c_1 = \frac{1}{4}n(n+1)$, this sequence of operators is related to L_b through *two* sequences of Darboux transformations. Define

$$L_n = \frac{1}{4}q_1^2\left(\partial_1^2 - \partial_2^2 + \frac{n(n+1)}{q_2^2}\right), \quad L_0 = L_b, \qquad \mathcal{D}_n^E = \partial_2 - \frac{n+1}{q_2},$$
$$\mathcal{D}_n^F = -2q_1q_2\partial_1 - (q_1^2 + q_2^2)\partial_2 - (n+1)\frac{(q_2^2 - q_1^2)}{q_2}.$$

These satisfy the *isospectral* Darboux relations

$$L_{n+1}\,\mathcal{D}_n^E = \mathcal{D}_n^E\,L_n, \qquad L_{n+1}\,\mathcal{D}_n^F = \mathcal{D}_n^F\,L_n, \qquad I_1^{(n+1)}\,\mathcal{D}_n^E = \mathcal{D}_n^E\,I_1^{(n)},$$
$$I_3^{(n+1)}\,\mathcal{D}_n^F = \mathcal{D}_n^F\,I_3^{(n)}.$$

The Darboux operators satisfy the following commutation relations

$$[\mathbf{H}, \mathcal{D}_n^E] = 2\mathcal{D}_n^E, \qquad [\mathbf{H}, \mathcal{D}_n^F] = 2\mathcal{D}_n^F, \qquad [\mathcal{D}_n^E, \mathcal{D}_n^F] = \mathbf{H}.$$

Remark 6.2. Since $u = -n(n+1)q_2^{-2}$ is a stationary solution of the higher KdV flows (starting with the flow of order $2n+1$), we can also build higher order operators which commute with both L_n and $I_1^{(n)}$. For instance, when $n = 1$, the third-order operator

$$I_4 = \partial_2^3 - \frac{3}{q_2^2}\partial_2 + \frac{3}{q_2^3}$$

has this property. This is Darboux related to $\mathbf{E}^3 = \partial_2^3$ through $\mathcal{D}_0^E$ and satisfies the relation $I_4^2 = I_1^3$. Under the involution, I_4 is related to an operator I_5, related to $\mathbf{F}^3$ through $\mathcal{D}_0^F$, and this commutes with both $I_3^{(1)}$ and L_1.

Example 6.5 (The quantum harmonic oscillator). Consider again case (41). Whilst this is not Darboux related to L_b for non-zero c_0, c_2, it *can be* so related to the quantum harmonic oscillator. We can adjust the values of c_0, c_2 by scaling transformations, whilst c_1 is derived from a Darboux transformation. Consider the operator

$$L_n = \frac{1}{4}q_1^2\left(\partial_1^2 - \partial_2^2 + 2(1-n) + q_2^2 - q_1^2 + \frac{n(n+1)}{q_2^2} - \frac{n(n+1)}{q_1^2}\right),$$

the last term just being an additive constant, which adjusts the eigenvalue. With these specific values of the parameters, we have commuting operators:

$$I_1^{(n)} = \partial_2^2 - 2(1-n) - q_2^2 - \frac{n(n+1)}{q_2^2},$$

$$I_2^{(n)} = 4(q_1\partial_1 + q_2\partial_2)^2 - 8(1-n)(q_2^2 - q_1^2) + 4(q_2^2 - q_1^2)^2.$$

The operators L_n and $I_1^{(n)}$ satisfy the following Darboux relations:

$$L_{n+1}\mathcal{D}_n = \mathcal{D}_n L_n, \qquad I_1^{(n+1)}\mathcal{D}_n = \mathcal{D}_n I_1^{(n)}, \quad \text{with} \quad \mathcal{D}_n = \partial_2 - q_2 - \frac{n+1}{q_2}.$$

When $n = 0$, the system (with zero eigenvalue) separates into a pair of *quantum harmonic oscillators*:

$$\left.\begin{array}{r}(\partial_1^2 - q_1^2)\varphi_1 = \mu\varphi_1,\\ (\partial_2^2 - q_2^2)\varphi_2 = (\mu+2)\varphi_2\end{array}\right\} \Rightarrow L_0(\varphi_1\varphi_2) = 0. \tag{42}$$

Using the standard *ladder operators*

$$A_i^{\pm} = \partial_i \pm q_i \quad \text{satisfying} \quad [\partial_i^2 - q_i^2, A_i^{\pm}] = \pm 2A_i^{\pm},$$

we have

$$(\partial_i^2 - q_i^2)\varphi_i = \lambda\varphi_i \Rightarrow (\partial_i^2 - q_i^2)(A_i^{\pm}\varphi_i) = (\lambda \pm 2)(A_i^{\pm}\varphi_i).$$

Starting with

$$\varphi_i^1 = e^{-(1/2)q_i^2}, \quad \text{satisfying} \quad A_i^+\varphi_i^1 = 0 \quad \text{and} \quad (\partial_i^2 - q_i^2)\varphi_i^1 = -\varphi_i^1,$$

we use A_i^- to build a sequence of eigenfunctions

$$\varphi_i^m = (A_i^-)^{m-1}\varphi_i^1 = (-1)^{m-1}H_{m-1}(q_i)e^{-(1/2)q_i^2},$$

$$\text{satisfying} \quad (\partial_i^2 - q_i^2)\varphi_i^m = (1-2m)\varphi_i^m,$$

where $H_m(z)$ is the Hermite polynomial of degree m. We then use (42) to build the sequence of eigenfunctions

$$\psi_{m,0} = \varphi_1^m \varphi_2^{m+1} = -H_{m-1}(q_1)H_m(q_2)e^{-(1/2)(q_1^2+q_2^2)} \quad \text{satisfying} \quad L_0\psi_{m,0} = 0.$$

The eigenfunctions of L_n are then built through the Darboux transformation

$$\psi_{m,n} = \mathcal{D}_{n-1}, \ldots, \mathcal{D}_0\psi_{m,0} \Rightarrow L_n\psi_{m,n} = \frac{1}{4}n(n+1)\psi_{m,n}.$$

The functions $\psi_{m,n}$ are also eigenfunctions of $I_1^{(n)}$, since this is isospectrally related to $I_1^{(0)}$, which is itself one of the one-dimensional harmonic oscillators. On the other hand, the operators $I_2^{(n)}$ are not related through the Darboux transformations, but do appear to act as $I_2^{(n)}\psi_{m,n} = \psi_{m+2,n} + \sum_{i=1}^m a_i\psi_{i,n}$, and thus as ladder operators.

7. Conclusions

In this paper we considered the Laplace–Beltrami operator of a two-dimensional constant curvature space, whose symmetry algebra is $sl(2, \mathbf{C})$. This high degree of symmetry is lost when potential functions are added and generally there would not exist any commuting operators. We used Darboux transformations to build potential functions for which there *do* exist commuting operators and, more importantly, whose spectrum and eigenfunctions can be *explicitly calculated*.

In this paper we only used *first-order* Darboux transformations, which automatically lead to *separable* potentials. However, the existence of Darboux transformations is much stronger than separability. The latter reduces a multi-dimensional Schrödinger equation to a collection of one-dimensional ones, but these are not themselves guaranteed to be *exactly solvable*, but *are* in the Darboux case. The Darboux transformation itself separates, with part of it leading to the Bäcklund chain for the KdV hierarchy. This meant that in the separated coordinates the potential functions were built out of the well known special solutions of the KdV hierarchy. We saw in the case of the Adler–Moser rational solutions (33), that the KdV equation is actually isospectral to our augmented Laplace–Beltrami operator. This phenomenon is not, of course, restricted to this particular solution of the KdV equation. *Any* solution of the KdV equation (or any equation in its hierarchy) can be used as a separated potential function. Whilst these are only $1 + 1$ dimensional equations, the spatial derivative can be taken in the direction of *any* of our Killing vectors, with a constraint along the other separation coordinate curve.

An interesting generalisation would be to use second or higher order Darboux transformations, which should lead to *non-separable* potentials. A further generalisation is to consider non-trivial *electromagnetic terms* (see, for example, [4,5]).

The approach is not restricted to two-dimensional spaces. The symmetry algebra of a higher dimensional space would generally be more interesting and the weight spaces would no longer be one-dimensional. Even first-order Darboux transformations could lead to non-separable potentials in higher dimensions.

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